QUANTUM AND CLASSICAL DESCRIPTION OF SPECTRAL MEASUREMENTS OF SIGNALS OF FREQUENCY STANDARDS

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ABSTRACT/ KEYWORDS

Classical and quantum descriptions of spectrum analyzer (SA) operation were suggested and the link between them was established for diffracting (grating) SA. It was shown that this SA transforms time statistics of photons into statistics of their space distribution at SA output plane. The link between quantum object, i.e. the signal of optical frequency standard, and SA interacting with it, was established. This is a contribution to the solution of global problem of quantum and classical physics joining for spectral measurements theory.

Spectrum, analyzer, classical, quantum, link.

INTRODUCTION

Theory of spectral measurements is a classical field of physics and technology, whereas principal problem of spectroscopy is the obtaining of function of photons distribution over frequency. Therefore a link between quantum nature of electromagnetic signals and SA operation has fundamental importance for spectroscopy.
In the paper [Ref.1] two aspects of this link were formulated and the results of researches about the establishment of link between quantum and classical description of electromagnetic signals of frequency standards are presented in the context of signal theory.
In this paper the principles of quantum theory of spectral measurements are developed, where a link between classical and quantum approaches is established. The basis of researches is classical input-output relation of SA for power spectral characteristics and physical signal theory principles [Ref.1], based on quantum representations. This link is established by strict calculations in the context of solution of basic problem of linear system (LS) and the introduction of complex apparatus spectrums in optical spectrometry.
The establishing of link between classical and quantum description of SA operation lies in the context of certain problems of quantum theory measurements. This is a fundamental problem of quantum and classical physics joining, the development of a measuring apparatus unified approach to quantum object and a interacting with it [Ref2]. This problem is not yet solved [Ref2], and this paper is devoted to its solving in the context of spectral measurements of electromagnetic signals of frequency standards.

CLASSICAL INPUT-OUTPUT RELATION OF LINEAR SYSTEM

For the measurements of complex spectrums SA is considered as LS. Generally, with the macro-description of LS, signal transformation is described by linear limited (continuous) operator $\widehat{A}$ in L_2 space :

$$\widehat{A} : X \to Y; x(\xi) \in X, y(\xi) \in Y, \xi \in \Re \quad (1)$$

where X, Y are linear functional spaces of inputs and outputs, respectively.
Basic characteristic of LS is its response to the input in the form $\delta(\xi - \xi')$. This response $Q(\xi, \xi') = \widehat{A}\delta(\xi - \xi')$, i.e. the action of operator $\widehat{A}$ to $\delta-$function, is defined as

$$\widehat{A}\,\delta(\xi - \xi') = \lim_{n\to\infty} \widehat{A}\,\delta_n(\xi, \xi') == \lim_{n\to\infty} Q_n(\xi, \xi'), \quad (2)$$

where $\delta_n(\cdot)$ is differentiable sequence of functions, convergent in the sense of distributions [Ref.3]:

$$\lim_{n\to\infty} \langle \delta_n(\xi, \xi'), x(\xi') \rangle = \langle \delta(\xi - \xi'), x(\xi') \rangle = \\ = x(\xi), x(\xi) \in A, \quad (3)$$

where A is a certain space of basic functions.
In general case $Q(\cdot)$ is distribution and is considered as the limit of sequence of linear functionals

$$\langle Q(\xi, \xi'), x(\xi') \rangle = \lim_{n\to\infty} \langle Q_n(\xi, \xi'), x(\xi') \rangle. \quad (4)$$

Function $\delta(\cdot)$ is representable by the series [Ref.3]

$$\delta(\xi - \xi') = \sum_{i=0}^{\infty} \psi_i(\xi)\psi_i(\xi'), \quad (5)$$

where $\{\psi_i\}$ is an orthonormal system of eigenfunctions of certain selfadjoint differential operator [Ref.3].
The series (5) converges in distribution space [Ref.3] and allows to choose $\delta_n(\cdot)$ in the form

$$\delta_n(\xi, \xi') = \alpha_n \sum_{i=0}^{n} \psi_i(\xi)\psi(\xi'), \quad (6)$$

where $\alpha_n \to 1, n \to \infty.$

The series of basic function $x(\cdot) \in L_2$ have the forms

$$x(\xi) = \sum_{i=0}^{\infty} a_i \psi_i(\xi); x_n(\xi) = \sum_{i=0}^{n} a_i \psi_i(\xi), \quad (7)$$

and $x(\cdot)$ converges in the space of basic functions.
The substitution of sum (6) in the functional of the right

side of Eq (4) at finite n gives

$$\left\langle \hat{A}\,\delta_n, x\right\rangle = \left\langle \hat{A}\,\alpha_n \sum_{i=0}^{n} \psi_i(\xi)\psi(\xi'), x(\xi)\right\rangle =$$

$$= \sum_{i=0}^{n} \alpha_n \left\langle \hat{A}\,\psi_i(\xi)\psi_i(\xi'), x(\xi')\right\rangle. \quad (8)$$

Because in Eq.(8) the operator $\hat{A}$ acts over variable ξ, and functional acts over variable ξ', that

$$\left\langle \hat{A}\,\delta_n, x\right\rangle = \sum_{i=0}^{n} \alpha_n\,\hat{A}\,\psi_i(\xi)\left\langle \psi_i(\xi'), x(\xi')\right\rangle =$$

$$= \alpha_n\,\hat{A} \sum_{i=0}^{n} a_i \psi_i(\xi) = \alpha_n\,\hat{A}\,x_n(\xi). \quad (9)$$

Since $\hat{A}$ is a linear limited operator in L_2, then [Ref.4]

$$\begin{aligned} &\|\hat{A}\,x_n - \hat{A}\,x\| = \|\hat{A}(x - x_n)\| \le \\ &\le \|A\|\cdot\|x_n - x\| \to 0, n \to \infty. \end{aligned} \quad (10)$$

Taking the limit in Eq (9) on the base of Eq (10) at $n \to \infty$ gives important relation

$$\left\langle \hat{A}\,\delta(\xi - \xi'), x(\xi')\right\rangle = \hat{A}\,x(\xi), \quad (11)$$

which states that linear limited operator commutates with convolution, if one of its cofactors is $\delta-$ function. Functional

$$\hat{A}\,x(\xi) = y(\xi) = \left\langle Q(\xi,\xi'), x(\xi')\right\rangle \quad (12)$$

expresses in the very general case the input-output relation for LS. If LS is such that functional (12) is regular, then it can be written as a linear integral operator

$$y(\xi) = \int_{\xi_1}^{\xi_2} Q(\xi,\xi')x(\xi')d\xi', \quad (13)$$

which plays a leading part in LS theory.

COMPLEX SPECTYRUMS MEASUREMENTS

For the measurements of complex harmonic spectrums Eq. (1) is written in the form

$$\hat{A}: \mathrm{S} \to \mathrm{S}_a; S(\omega) \in \mathrm{S}, S_a(\omega) \in \mathrm{S}_a, \quad (14)$$

where S, S_a are linear spaces of input (mathematical) and apparatus spectrums, respectively; ω is a temporal spectral frequency; $S(\omega) = \hat{F}s(t)$, $\hat{F}$ is the operator of Fourier transformation; $s(t)$ is an analyzed oscillation as a function of time t.

According to (14), Eq. (13) is written as

$$S_a(\omega) = \int_{\omega_1}^{\omega_2} K(\omega,\omega')S(\omega')d\omega', \quad (15)$$

where $K(\cdot)$ is complex spectrum spread function (SSF), i.e. SA response to the function $\delta(\omega - \omega')$.

Because $\delta(\omega - \omega') = \hat{F}\exp(i\omega' t)$, then complex SSF is the result of action of $s(t) = \exp(i\omega' t)$ to SA.

As real information system, SA can operate with limited information volume. This means that at spectral measurements any supported oscillation $s(t)$ must has bounded degrees of freedom $W = \Delta\omega T/\pi$, where $2\Delta\omega$ is analyzed bandwidth, T is analysis duration. Basic metrological characteristics $2\Delta\omega$ and T of SA are established by truncation operations.

The first of them is the operation $\hat{B}$ of spectrum truncation

$$\hat{B}\,s(t) = \frac{1}{2\pi}\int_{\omega_0-\Delta\omega}^{\omega_0+\Delta\omega} \exp(i\omega t)S(\omega)d\omega, \quad (16)$$

where ω_0 is center of analyzed bandwidth.

The second operation is the operation $\hat{D}$ of time function $\hat{B}\,s(t)$ truncation:

$$\hat{D}\hat{B}\,s(t) = \chi(t)\,\hat{B}\,s(t), \quad (17)$$

where

$$\chi(t) = \begin{cases} 1, t \in (-T/2, T/2) \\ 0, t \notin (-T/2, T/2) \end{cases}. \quad (18)$$

The third operation is the operation $\hat{M}$ of measurement of complex spectrum $S_a(\omega)$

$$\hat{M}\,\hat{D}\hat{B}\,s(t) = \int_{\omega_1}^{\omega_2} K_{\mathrm{M}}(\omega,\omega')\hat{F}\chi(t)\hat{B}s(t)d\omega' =$$

$$= S_a(\omega) = \int_{\omega_0-\Delta\omega}^{\omega_0+\Delta\omega} K(\omega,\omega')S(\omega')d\omega', \quad (19)$$

where $K_{\mathrm{M}}(\cdot)-$ is complex SSF of real SA, fulfilling the operation $\hat{M}$;

$$K(\omega,\omega') = \int_{\omega_1}^{\omega_2} K_{\mathrm{M}}(\omega,\omega'')\cdot$$

$$\cdot\,\mathrm{sinc}[(\omega' - \omega'')T/2]d\omega'', \quad (20)$$

where $\mathrm{sinc}(\cdot)$ is complex SSF of "real SA in perfect performance", which fulfills truncated Fourier transformation $\hat{F}\hat{D}\hat{B}s(t)$.

Eq. (19) describes simple measurement of complex spectrum, when the onset of time reference $t = 0$. In real condition, the dynamics of SA interaction with analyzed oscillation $s(t)$ is such that apparatus spectrums vary over time. These spectrums are introduced with the help of translation operation $\hat{T}$

$$\hat{T}g(t') = g(t' \pm t), \quad (21)$$

acting either on the function $s(t)$ or $\chi(t)$ in Eq. (19).

Denumerable set $\{t_n\}$ defines sampling spectrums $S(\omega, t_n)$, if set $\{t\}$ is continuum, then instantaneous complex spectrums (ICS) $S(\omega, t)$ are measured.

The spectral measurements of microwave frequency standards signals are realized by radiospectrometers, which measure sampling spectrums of oscillating processes. In this case the operator $\widehat{n}$ acts on function $\chi x t i$.

Furthermore, ICS measurements are considered at the action of operator $\widehat{n}$ on function $s x t i$, i.e. on condition that oscillation $s x t i$ is extended by the window $\chi x t i$. In this case SA operation is described as

$$S_a x\omega d t i = \int_{\omega_a - \Delta\omega}^{\omega_a + \Delta\omega} K x\omega d\omega' i \, L Q y x - i\omega' t i \cdot$$

$$\cdot \, S x\omega' i d\omega' a \qquad (22)$$

In Eq. (22) kernel, i.e. complex SSF, in the form $K x\omega d\omega' i \, L Q y x - i\omega' t i$ is necessary and sufficient condition of ICS measurement.. This measurement is described by the operator $\widehat{T} = \widehat{K}\widehat{s}\,\widehat{t}\,\widehat{n}\widehat{S}$, where $\widehat{n}\widehat{S} = \widehat{S}\widehat{n}$.

POWER SPECTRUM ESTIMATION

The estimation of power spectrum $\overline{G} x\omega i$ includes the operation $\widehat{W}$ of function $| S_a x\omega d t i |^g$ calculation and the operation $\widehat{\mathrm{I}}$ of its time averaging over time T_R, i.e.

$$\overline{G} x\omega i = \widehat{\mathrm{I}}\widehat{W} S_a x\omega i = \int_{-T_R}^{T_R} S_a x\omega d t i \, S_a^* x\omega d t i \, dt. \quad (23)$$

Thus, the estimation of energy distribution over frequency on the base of ICS measurement is described by the following nodal operations:

$$\overline{G} x\omega i = \widehat{\mathrm{I}}\widehat{W}\widehat{K}\widehat{s}\,\widehat{t}\,\widehat{n}\widehat{S} s x t i = \widehat{\mathrm{I}}\widehat{W}\widehat{T} s x t i. \qquad (24)$$

The substitution of $S_a x\omega d t i$ in the form (22) in Eq. (23), transfer to dimensionless variables and integration by time give [Ref.6]

$$\overline{G} x\eta i = C \int_{-1}^{1} d\eta' \int_{-1}^{1} K x\eta d\eta' i K^* x\eta d\eta'' i \cdot$$

$$\cdot \, F x\eta' i F^* x\eta'' i \, \mathrm{sinc}[c x\eta' - \eta'' i] d\eta'' a \; (25)$$

where $C = const$, $F x \cdot i$ is the spectrum of complex envelope of signal $s x t i$; $c = \Delta\omega T_R >> 1$.

Function $K x \cdot i F x \cdot i$ can be represented by series over the system of prolate spheroidal wave functions [Ref.5]

$$K x\eta d\eta' i F x\eta' i = \sum_{i=a}^{N} a_i x\eta i \psi_i x\eta' i d \qquad (26)$$

The substitution of sum (26) in Eq. (25) and the application of methods of theory of these functions give

$$\int_{-1}^{1} K x\eta d\eta' i F x\eta' i \, \mathrm{sinc}[c x\eta' - \eta'' i] d\eta' =$$

$$= \sum_{i=a}^{n} a_i x\eta i \int_{-1}^{1} \psi_i x\eta' i \, \mathrm{sinc}[c x\eta' - \eta'' i] d\eta' =$$

$$= \sum_{i=a}^{N} a_i x\eta i \psi_i x\eta'' i \lambda_j \approx K x\eta d\eta'' i F x\eta'' i a \; (27)$$

where $\lambda_i \approx 1$, if $c >> 1$.

The result (27) allows to write Eq. (25) as [Ref.6]

$$\overline{G} x\eta i = C \int_{-1}^{1} | K x\eta d\eta' i |^g \cdot | F x\eta' i |^g \, d\eta' a \; (28)$$

where return to natural variables gives familiar in optical spectrometry linear integral operator, which establishes the SA input-output relation for energy distribution over frequency

$$\overline{G} x\omega i = C_R \int_{\omega_1}^{\omega_g} R x\omega d\omega' i G x\omega' i d\omega' a \qquad (29)$$

where $C_R = const$; $R x \cdot i = | K x \cdot i |^g$ is power SSF; $G x \cdot i = | S x \cdot i |$ is power spectrum of oscillation $s x t i$, $t \in x - T_R / g d T_R / g i$.

SPECTRAL MEASUREMENTS BY DIFFRACTING SPECTRUM ANAYZERS

Optical frequency standards signals are wave processes, generated by source oscillations, their relation to the uniform plane wave, falling on SA, is given by the operator $\widehat{\mathrm{W}}$:

$$s x c_a t - z i = \widehat{\mathrm{W}} s x t i d \qquad (30)$$

where operator $\widehat{\mathrm{W}}$ differs from operator $\widehat{n}$ only by dimension scale.

Diffracting SA includes the following idealized basic units:

1. Grating, having transparency function

$$P x x i = L Q y x i \Omega_g x i d \qquad (31)$$

where Ω_g is spatial frequency; x is one of Cartesian coordinates, in which SA is considered.

2. Optical Fourier-processor, including two layers of free space and lens with focal length f between them, where kernel fulfilling Fourier transformation is

$$R x x' d\omega_l' d x i = L Q y x - i\omega_l' x' x / c_a f i a \qquad (32)$$

where ω_l' is circular frequency of monochromatic optical radiation; x' is Cartesian coordinate at SA output plane; c_a is light velocity;

3. Square photodetector with time integrator.

If uniform plane monochromatic light wave falls on SA input, then light field on its output plane is described as

$$S(t, K', \omega'_a) = i_S \exp(s\omega'_d t) \cdot$$
$$\cdot \int_{-G_d/2}^{G_d/2} C(K)T(K', \omega'_a, K)RK, \quad (33)$$

where $i_S = cFont$; G_d is aperture dimension of SA. The substitution of (31) and (32) in Eq. (33) gives

$$S(t, K', \omega'_a) = 2i_S \exp(s\omega'_d t) \cdot$$
$$\cdot \mathrm{sinc}\{[(\Omega_j \not{N}_0 / K') - \omega'_a]G_d K' / 2\not{N}_0\}. \quad (34)$$

The comparison of Eqs. (34) and (22) shows that the function $S(\cdot)$ is complex SSF of SA, fulfilling ICS measurement of oscillation $n(t)$, which generates uniform plane wave [Ref.6]. Really, volume $\Omega_j \not{N}_0 / K' = \omega(K')$ is temporal spectral frequency, and volume $G_d K' / \not{N}_0 = z\,(K')$ is analysis time duration [Ref.6], that follows from comparison of function $\mathrm{sinc}(\cdot)$ in Eq. (34) with SSF of "actual SA in perfect performance", noted in Eq. (20).

Then light oscillations at output plane of diffracting SA are detected by squaring photodetector and are integrated over time. These operations result is the estimation (29) of power spectrum of source oscillations $n(t)$. Thus, power spectrum estimation by diffracting SA is described by the following product of operators

$$\overline{x}\,(\omega) = \hat{\mathrm{I}}\hat{\mathrm{Q}}\hat{\mathrm{M}}\hat{\mathrm{G}}\hat{\mathrm{D}}\hat{\mathrm{B}}\hat{\mathrm{W}}n(t), \hat{\mathrm{B}}\hat{\mathrm{W}} = \hat{\mathrm{W}}\hat{\mathrm{B}}, \quad (35)$$

where $\hat{\mathrm{G}}$ is the operator of multiplication by the function $C(K)$; $\hat{\mathrm{D}}$ is the aperture truncation operator; operators product $\hat{\mathrm{M}}\hat{\mathrm{G}}\hat{\mathrm{D}}\hat{\mathrm{W}}\hat{\mathrm{B}} = \hat{\mathrm{H}}$ describes ICS measurement.

Function $n(t)$ is a classical object of processing (35), their result is mapping of time variations, i.e. sources oscillations, into space characteristics of distribution over frequencies (35). This processing can be applied formally to function, which describes time statistics photons, composing analyzed signals. That function $\Phi(t)$ is introduced by the following reasoning.

At the conditions of classical description of electromagnetic field [Refs1, 7] the expression of classical instantaneous power almost certainty describes in time the distribution of instants of photons birth, i.e. its time statistics or birth statistics [Ref.7]. Therefore, that similar and proportional to function $n(t)$ is function

$$\Phi(t) = \sqrt{P(t)} \cdot \mathrm{spin}(t), \quad (36)$$

where $P(t) = n^2(t)$ is classical instantaneous power; $\mathrm{spin}(t)$ is the function, describing the orientation of photons spins at quantum description of electromagnetic signal [Refs 1, 7].

It follows from Eq. (36) that sense of function $\Phi^2(t)$ completely corresponds to probability sense, which is assigned to the squared modulus of electromagnetic field oscillations in quantum physics. The transformation of function $\Phi(t)$ by the law (35) gives the statistics of space distribution of photons (with account for spins distribution) at the output plane of diffracting SA. This statistics corresponds to the time statistics of photons at diffracting SA input. In the end the transformation of these statistics is given by operator product (35), where all operators have mathematical sense only.

The transformation of function (36) by the law (35) shows that the operation of diffracting SA can be interpreted as the transformation of time statistics of photons, composing analyzed signal, into the statistics of their space distribution at the output plane of this SA.

CONCLUSION

In this paper the following problems are solved:

1. Through strict mathematical calculations the problem of LS input-output relation was solved, when LS is considered as a "black box". This solution is more general than the solution of the problem of representation of linear operator in integral form, because linear integral operator is the special case of linear functional, obtained here. Relations, describing the measurements of complex and power spectrums, were obtained on this basis. In this connection ICS, unknown in the theory of optical spectral measurements, were established.
2. The link between classical and quantum descriptions of diffracting SA operation was established. It was shown that diffracting SA transforms time statistics of photons into statistics of their space distribution at the output plane. This poses the question about the interpretation of Davisson and Garmer experiment without resort to De Broglie waves. The case in point is interpretation of this experiment as a transformation of time statistics of electrons into statistics of their space distribution.

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